16.1 Lecture: Line integrals

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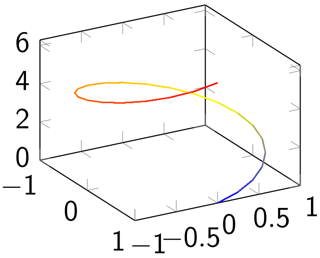
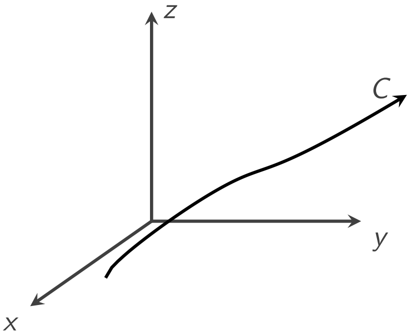
Spring 2019

Space curves



Recall from Chapter 13 that we have spent some time working with space curves and vector-valued functions. The picture above is our old friend the helix. Chapter 16 now returns us to similar topics.

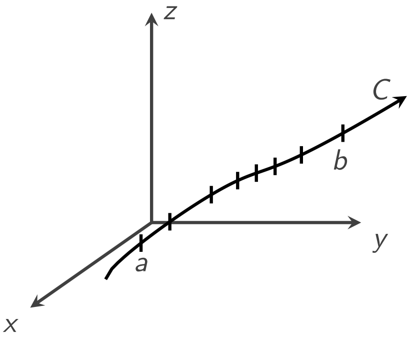
Line integrals



A line integral is the integral of a function $F(x, y, z)$ along a curve C. The next few slides are dedicating to investigating what that means.

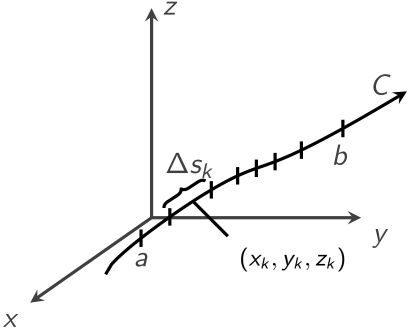
Line integrals

If we set things up formally, integrating a function along a curve would mean we want to pick a place to begin, say $t = a$, and a place to end, say $t = b$, and break the curve up between those points into many small pieces.



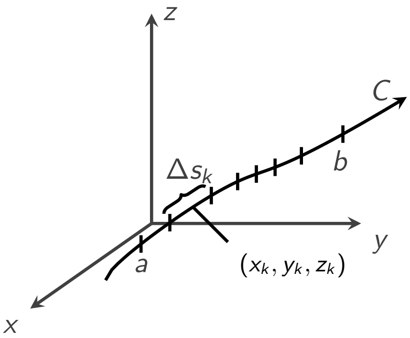
Line integrals

If we have n small pieces and we call the length of the k^{th} small piece Δs_k, then we can estimate the integral of $F(x, y, z)$ along the curve C by choosing a point (x_k, y_k, z_k) on the k^{th} small piece and multiplying the $F(x_k, y_k, z_k)$ by Δs_k.



Line integrals

If we add $F(x_k, y_k, z_k)\Delta s_k$ over each small piece, we have an approximation for the integral.



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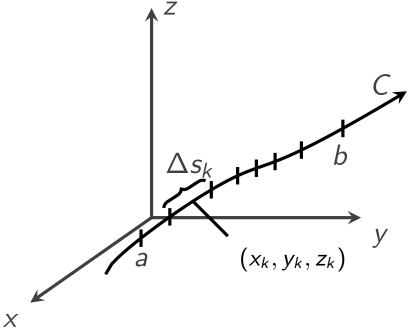

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta s_k$$

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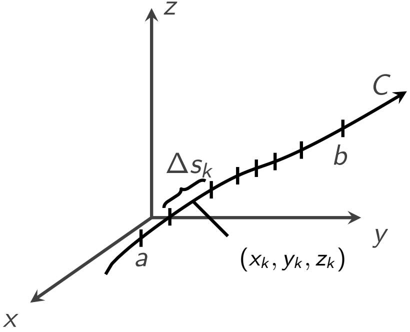

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta s_k$$

We define the line integral to be the limit of S_n as n goes to infinity.

Line integrals

Definition

If $F(x, y, z)$ is defined on a curve C given parametrically by $\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}$, $a \leq t \leq b$, then the line integral of F over C is

$$\int_C F(x, y, z)ds = \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta s_k$$

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Evaluating line integrals

To determine how to evaluate a line integral, we have to do two things: First, we must express what it means for the function $F(x, y, z)$ to be on the curve C.

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This is fairly straight-forward, since the curve is given by a parametrization $\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}$, so we can plug what x, y, z are into the function:

$$F(f(t), g(t), h(t))$$

Evaluating line integrals

Second, we must determine what ds is in terms of t. This requires us to recall the formula for the arclength parameter with basepoint a:

$$s(t) = \int_a^t \|\vec{\mathbf{v}}(\tau)\| d\tau$$

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Thus, by the fundamental theorem of calculus, we have

$$\frac{ds}{dt} = \|\vec{\mathbf{v}}(t)\| \text{ or } ds = \|\vec{\mathbf{v}}(t)\| dt.$$

Evaluating line integrals

Putting it all together, we can replace

$$\int_C \text{ by } \int_{t=a}^{t=b}$$

we can replace $F(x, y, z)$ by $F(f(t), g(t), h(t))$, and ds by $\|\vec{\mathbf{v}}(t)\|dt$. So

$$\int_C F(x, y, z)ds = \int_{t=a}^{t=b} F(f(t), g(t), h(t))\|\vec{\mathbf{v}}(t)\|dt.$$

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$$\int_C F(x, y, z)ds = \int_{t=a}^{t=b} F(f(t), g(t), h(t))\|\vec{\mathbf{v}}(t)\|dt.$$

Thus to calculate a line integral over a curve C, we must find a smooth parametrization $\vec{\mathbf{r}}(t)$ of C and then use the formula above.

Example

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Integrate $F(x, y, z) = x - 3y^2 + z$ *over the line* C *joining* $(0, 0, 0)$ *to* $(1, 1, 1)$.

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We need to parametrize the line described in the problem. That just means we have to find a way to write the line down as a vector-valued function of t. In general, we can parametrize the line segment from P_1 to P_2 with the vector function

$$\vec{\mathbf{r}}(t) = P_1(1 - t) + P_2(t), \quad 0 \leq t \leq 1.$$

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In this case, that means C is parametrized by
$\vec{\mathbf{r}}(t) = \langle 0, 0, 0, \rangle(1 - t) + \langle 1, 1, 1\rangle(t) = \langle t, t, t\rangle, \quad 0 \leq t \leq 1.$

Example, cont.

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Next we have to calculate $\|\vec{\mathbf{v}}(t)\|$, plug $\vec{\mathbf{r}}(t)$ into $F(x, y, z)$, and calculate the line integral.

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$$\|\vec{\mathbf{v}}(t)\| = \|\langle 1, 1, 1 \rangle\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$F(t, t, t) = t - 3t^2 + t = 2t - 3t^2$$

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Thus the line integral is

$$\int_C (x - 3y^2 + z)ds = \int_{t=0}^{t=1} (2t - 3t^2)(\sqrt{3})dt = \sqrt{3}\Big[t^2 - t^3\Big]_0^1 = 0$$

A note of caution

For regular integrals, the fundamental theorem of calculus tells us that the integral only depends on the endpoints of the integral, since $\int_{t=a}^{t=b} f(t)dt = F(b) - F(a)$ where F is an antiderivative of f. However, this is not the case with line integrals. For example, we could ask this question:

Example

Integrate $F(x, y, z) = x - 3y^2 + z$ *over the curve consisting of the line from* $(0, 0, 0)$ *to* $(1, 1, 0)$ *and then the line from* $(1, 1, 0)$ *to* $(1, 1, 1)$.

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We will get a different answer than we got for the previous question.

Example 2

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The two lines are parametrized as

$$\vec{\mathbf{r}}_1(t) = \langle 0, 0, 0\rangle(1 - t) + \langle 1, 1, 0\rangle(t) = \langle t, t, 0\rangle, \quad 0 \leq t \leq 1$$

and

$$\vec{\mathbf{r}}_2(t) = \langle 1, 1, 0\rangle(1 - t) + \langle 1, 1, 1\rangle(t) = \langle 1, 1, t\rangle, \quad 0 \leq t \leq 1.$$

We calculate

$$\|\vec{\mathbf{v}}_1(t)\| = \sqrt{1 + 1} = \sqrt{2} \text{ and } \|\vec{\mathbf{v}}_2(t)\| = \sqrt{1} = 1.$$

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$$\|\vec{\mathbf{v}}_1(t)\| = \sqrt{1+1} = \sqrt{2} \text{ and } \|\vec{\mathbf{v}}_2(t)\| = \sqrt{1} = 1.$$

Thus

$$\int_C (x - 3y^2 + z)ds = \int_0^1 F(t, t, 0)\sqrt{2}dt + \int_0^1 F(1, 1, t)dt$$

$$= \sqrt{2}\int_0^1 (t - 3t^2)dt + \int_0^1 (1 + 3 - t)dt = \left[\frac{\sqrt{2}t^2}{2} - \sqrt{2}t^3 + \frac{t^2}{2} - 2t\right]_0^1$$

$$= -\sqrt{2}/2 - 3/2.$$